EXERCISE – V

HINTS & SOLUTIONS

Sol.1 **(a)** $I = \int_0^{\pi/2} \frac{\cos^9 x}{\cos^3 x + \sin^3 x} dx$

$I = \int_0^{\pi/2} \frac{\sin^9 x}{\cos^3 x + \sin^3 x} dx$

$2I = \int_0^{\pi/2} \frac{\cos^9 x + \sin^9 x}{\cos^3 x + \sin^3 x} dx$

$2I = \int_0^{\pi/2} (\cos^6 x + \sin^6 x + 2\sin^3 x \cos^3 x)\, dx$

(b) $I = \int_0^{\pi} \frac{x\,dx}{1+\cos\alpha \sin x}$

King

$I = \int_0^{\pi} \frac{(\pi - x)\,dx}{1+\cos\alpha \sin x}$

$2I = \pi \int_0^{\pi} \frac{dx}{1+\cos\alpha \sin x}$

$I = \frac{\pi}{2} \int_0^{\pi} \frac{\sec^2 \frac{x}{2}}{1+\tan^2\frac{x}{2} + 2\cos\alpha \tan\frac{x}{2}}$

Put $\tan\frac{x}{2} = t$ $\Rightarrow 1/2\sec^2\frac{x}{2} dx = dt$

$= \pi \int_0^{\infty} \frac{dt}{1+t^2+2t\cos\alpha}$

$= \pi \int_0^{\infty} \frac{dt}{t^2+2t\cos\alpha + \cos^2\alpha \sin^2\alpha}$

$= \pi \int_0^{\infty} \frac{dt}{(t+\cos\alpha)^2 + \sin^2\alpha}$

$= \frac{\pi}{\sin\alpha}\left[\tan^{-1}\frac{t+\cos\alpha}{\sin\alpha}\Big|_0^{\infty}\right]$

$= \frac{\pi}{\sin\alpha}\left[\frac{\pi}{2} - \tan^{-1}(\cot\alpha)\right]$

$= \frac{\pi}{\sin\alpha}\left[\frac{\pi}{2} - \tan^{-1}\tan\left(\frac{\pi}{2} - \alpha\right)\right]$

$= \frac{\pi\alpha}{\sin\alpha}$ if $a \in (0, \pi)$

$= \frac{\pi}{\sin\alpha}(\alpha - 2\pi)$ if $\alpha \in (\pi, 2\pi)$

Sol.2 **(a)** $f(x) = \int_1^{x} \sqrt{2-t^2}\, dt$

$f'(x) = \sqrt{2-x^2}$

$x^2 - f'(x) = 0$

$x^2 = \sqrt{2-x^2}$

$x^4 = 2 - x^2$

$x^4 + x^2 - 2 = 0$

$x^4 + 2x^2 - x^2 - 2 = 0$

$x^2(x^2+2) - 1(x^2+2) = 0$

$x^2 = -2,\ x^2 = 1 \Rightarrow x = \pm 1$

(b) $I = \int_3^{3+3T} f(2x)\, dx = 3\int_0^{T} f(2x)\, dx$

$= \frac{3}{2}\int_0^{2T} f(x)\,dx = 3\int_0^{T} f(x)\,dx = 3I$

(c) $I = \int_{-1/2}^{1/2} [x]\,dx + \underbrace{\int_{-1/2}^{1/2} \ell n\left(\frac{1+x}{1-x}\right) dx}_{\text{as odd function}}$

$= \int_{-1/2}^{0} (-1)\,dx + \int_0^{1/2} 0.dx$

$= (-1)\left(0 + \frac{1}{2}\right) = \frac{1}{2}$

Sol.3 $\int_0^{\pi/2} f(\cos 2x)\cos x\, dx = \sqrt{2}\int_0^{\pi/4} f(\sin 2x)\cos x\, dx$

King

$I = \int_0^{\pi/2} f(\cos 2x)\sin x\, dx$

$$2I = \int_0^{\pi/2} f(\cos 2x)\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right)dx$$

$$2I = \sqrt{2}\int_0^{\pi/2} f(\cos 2x)\cos\left(\frac{\pi}{4} - x\right)dx$$

$$2I = -\sqrt{2}\int_{-\pi/4}^{\pi/4} f(\sin 2x)\cos x\, dx$$

$$I = \sqrt{2}\int_0^{\pi/4} f(\sin 2x)\cos x\, dx$$

Sol.4 **(a)** $I = \int_0^1 \sqrt{\frac{1-x}{1+x}}\,dx$

$$= \int_0^1 \frac{1-x}{\sqrt{1-x^2}}\,dx$$

$$= \sin^{-1} x|_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= \frac{\pi}{2} - \frac{1}{2}\int_0^1 \frac{dt}{\sqrt{1-t}} = \frac{\pi}{2} - 1$$

(b) $\int_0^{t^2} x f(x)\,dx = \frac{2}{5}t^5$

$t^2 f(t^2) \cdot 2t = 2t^4$

$f(t^2) = t$

Put $t = \frac{2}{5}$

$f\left(\frac{4}{25}\right) = \frac{2}{5}$

(c) $y(x) = \int_{\pi^2/16}^{x^2} \frac{\cos x.\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}\,d\theta$

$$\frac{dy}{dx} = \frac{\cos x \cos x}{1+\sin^2 x}(2x)$$

at $x = \pi$

$$\frac{dy}{dx} = 2\pi$$

(d) $I = \int_{-\pi/3}^{\pi/3} \frac{\pi}{2-\cos\left(|x|+\frac{\pi}{3}\right)}dx + \underbrace{\int_{-\pi/3}^{\pi/3} \frac{4x^3 dx}{2-\cos\left(|x|+\frac{\pi}{3}\right)}}_{\text{as odd function}}$

$$= \int_{-\pi/3}^{\pi/3} \frac{\pi\, dx}{2-\cos\left(|x|+\frac{\pi}{3}\right)}$$

$$= 2\int_0^{\pi/3} \frac{\pi\, dx}{2-\cos\left(x+\frac{\pi}{3}\right)} \quad \text{put } x + \frac{\pi}{3} = t$$

$$= 2\pi\int_{\pi/3}^{2\pi/3} \frac{dt}{2-\cos t} = 2\pi\int_{\pi/3}^{2\pi/3} \frac{\sec^2 t/2}{1+3\tan^2\frac{t}{2}}\,dt$$

$$= 2\pi\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{2dz}{1+3z^2} = \frac{4\pi}{3}\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{dz}{z^2+\left(\frac{1}{\sqrt{3}}\right)^2}$$

$$I = \frac{4\pi}{3}\sqrt{3}\,[\tan^{-1}\sqrt{3}\,z]_{1/\sqrt{3}}^{\sqrt{3}}$$

$$= \frac{4\pi}{\sqrt{3}}\left[\tan^{-1}3 - \frac{\pi}{4}\right]$$

$$= \frac{4\pi}{\sqrt{3}}\tan^{-1}\left(\frac{1}{2}\right)$$

Sol.5 **(a)** $\int_{\sin x}^{1} t^2 f(t)\,dt = (1 - \sin x)$

$-\sin^2 x\, f(\sin x)\cos x = \cos x$

$f(\sin x) = \frac{1}{\sin^2 x}$

$f\left(\frac{1}{\sqrt{3}}\right) = 3$

(b) $I = \int_{-2}^{0} (x^3 + 3x^2 + 3x + 3 + (x+1)\cos(x+1))\,dx$

Put $x + 1 = t$

$$= \int_{-1}^{1} [(t^3 + t\cos t) + 2]\,dt = \int_{-1}^{1} 2\,dt = 4$$

(c) $I = \int_0^{\pi} e^{|\cos x|}\left(2\sin\left(\frac{1}{2}\cos x\right) + 3\cos\left(\frac{1}{2}\cos x\right)\right)\sin x\,dx$ x

$\downarrow$

0

(By using Queen)

$$= 6\int_0^{\pi/2} e^{\cos x}\sin x\cos\left(\frac{1}{2}\cos x\right)dx$$

Put cosx = t $\Rightarrow$ sin x dx = – dt

$$= -6\int_1^0 e^t\cos\left(\frac{t}{2}\right)dt$$

$a = 1$

$b = \frac{1}{2}$

$$\frac{e^t}{1+\frac{1}{4}}\left[\cos\frac{t}{2} + \frac{1}{2}\sin\frac{t}{2}\right]_0^1$$

$$= \frac{24}{5}\left[e\cos\left(\frac{1}{2}\right) + \frac{e}{2}\sin\left(\frac{1}{2}\right) - 1\right]$$

Sol.6 (a) $\int_0^{\pi/2}\sin x\,dx = \frac{\frac{\pi}{2}+0}{4}\left[\sin(0) + \sin\left(\frac{\pi}{2}\right) + 2\sin\left(\frac{0+\frac{\pi}{2}}{2}\right)\right]$

$$= \frac{\pi}{8}(1+\sqrt{2})$$

(b) $\lim_{t\to a}\left\{\frac{\int_0^t f(x)\,dx - \left(\frac{t-a}{2}\right)(f(t)+f(a))}{(t-a)^3}\right\} = 0$

use L-hospital rule

$$\Rightarrow \lim_{t\to a}\left\{\frac{f(t) - \left(\frac{t-a}{2}\right)f'(t) - \frac{1}{2}(f(t)+f(a))}{3(t-a)^2}\right\} = 0$$

$$\Rightarrow \lim_{t\to a}\left\{\frac{\frac{1}{2}\{f(t)-f(a)\} - \frac{1}{2}(t-a)f'(t)}{3(t-a)^2}\right\} = 0$$

$$\Rightarrow \lim_{t\to a}\left\{\frac{\frac{1}{2}f'(t) - \frac{1}{2}(t-a)f''(t) - \frac{1}{2}f'(t)}{6(t-a)}\right\} = 0$$

$$\Rightarrow \lim_{t\to a} -\frac{1}{12}\frac{(t-a)f''(t)}{(t-a)} = 0$$

$\Rightarrow$ f" (a) = 0 $\Rightarrow$ degree of polynomial f(x) is '1'

(c) $F'(c) = (b-a)f'(c) + f(a) + f(b)$

$F'' = f''(c)(b-a) < 0$

$F'(c) \neq 0$ $\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$

Sol.7 $I = \frac{5050\int_0^1 (1-x^{50})^{100}dx}{\int_0^1 (1-x^{50})^{101}dx}$

$$= 5050\,\frac{I_{100}}{I_{101}}$$

$$I_{101} = \int_0^1 (1+x^{50})(1-x^{50})^{100}\,dx$$

$$= I_{100} - \left[\frac{-x(1-x^{50})^{101}}{101}\right]_0^1 - \int_0^1\frac{(1-x^{50})^{101}}{5050}$$

$$I_{101} = I_{100} - \frac{I_{101}}{5050}$$

$$5050\,\frac{I_{100}}{I_{101}} = 5051$$

Sol.8 **(a)** $\lim_{x\to\frac{\pi}{4}}\frac{\int_2^{\sec^2 x} f(t)\,dt}{x^2 - \frac{\pi^2}{16}}$ Using leibnits

$$\lim_{x\to\frac{\pi}{4}}\frac{f(\sec^2 x)x\sec x.\tan x\sec x}{x}$$

$$= \frac{f\left(\sec^2\frac{\pi}{4}\right)\sec^2\left(\frac{\pi}{4}\right)\tan\frac{\pi}{4}}{\frac{\pi}{4}} = \frac{8}{\pi}f(2)$$

(b) **(A)** $\int_{-1}^{1}\frac{dx}{1+x^2} = 2\int_0^1\frac{dx}{(1+x^2)} = 2\tan^{-1}x\,|_0^1 = \frac{\pi}{2}$

(B) $I = \int_0^1\frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x\,|_0^1 = \frac{\pi}{2}$

(C) $I = \int_2^3\frac{dx}{1-x^2} = \frac{1}{2}\ \ell n\ \frac{2}{3}$

(D) $\int_1^2\frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}x|_1^2$

$= \sec^{-1}2 - \sec^{-1}1$

$= \frac{\pi}{3}$

Sol.9 $S_n < \lim_{n\to\infty} S_n = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\frac{1}{1+\left(\frac{k}{n}\right)+\left(\frac{k}{n}\right)^2}$

$= \int_0^1\frac{dx}{1+x+x^2} = \frac{\pi}{3\sqrt{3}}$

$T_n > \frac{\pi}{3\sqrt{3}}$

as $h\sum_{k=0}^{n-1}f(kh) > \int_0^1 f(x)\,dx > h\sum_{k=1}^{n}f(kh)$

Sol.10 **(a)** $\int_0^x\sqrt{1-(f'(t))^2}\ dt = \int_0^x f(t)\,dt$

$f'(x) = \sqrt{1-f^2(x)} \qquad \Rightarrow f(x) = \sin x$

Also $x > \sin x$ for $x = 0$

(b) $I_n = \int_{-\pi}^{\pi}\frac{\sin nx}{(1+\pi^x)\sin x}dx$

Use king

$= \int_{-\pi}^{\pi}\left(\frac{\sin nx}{(1+\pi^x)\sin x}+\frac{\pi^x\sin nx}{(1+\pi^x)\sin x}\right)dx$

$I_n = \int_{-\pi}^{\pi}\frac{\sin nx}{\sin x}dx = 2\int_0^{\pi}\frac{\sin nx}{\sin x}dx$

$I_{n+2} - I_n = 2\int_0^{\pi}\frac{\sin(n+2)x-\sin nx}{\sin x}dx$

$I_{n+2} - I_n = 2\int_0^{\pi}\frac{2\cos(n+1)x.\sin x}{\sin x}dx = 0$

$I_{n+2} = I_n$

$I_1 = \pi,\ \ I_2 = \int_0^{\pi}2\cos x\,dx = 0$

(c) $f(x) = \int_0^x f(t)\,dt \Rightarrow f(0) = 0$

$f'(x) = f(x)$

$\frac{f'(x)}{f(x)} = 1$

$\ell n\ f(x) = x + C$

$f(x) = Ke^x$

$k = 0$

$f(x) = 0$

$f(\ell n5) = 0$

Sol.11 **(a)** $\lim_{x\to 0}\frac{1}{x^3}\int_0^x\frac{t\,\ell n(1+t)}{t^4+4}\,dt$

$= \lim_{x\to 0}\frac{x\,\ell n(1+x)}{3x^2(x^4+4)}$

$= \lim_{x\to 0}\frac{1}{3}\left(\frac{\ell n(1+x)}{x}\right)\frac{1}{(x^4+4)} = \frac{1}{12}$

(b) $I = \int_0^1\frac{x^4(1-x)^4}{1+x^2}dx$

$= \int_0^1\left(x^6-4x^5+5x^4-4x^2+4-\frac{4}{1+x^2}\right)dx$

$= \left[\frac{x^7}{7}-\frac{2x^2}{3}+x^5-\frac{4x^3}{3}+4x\right]_0^1 - \pi$

$= \frac{1}{7} - \frac{2}{3} + 1 - \frac{4}{3} + 4 - \pi = \frac{22}{7} - \pi$

(c) $e^{-x}f(x) = 2 + \int_0^x\sqrt{t^4+1}\ dt \quad ....(1)$

$f(f^{-1}(x)) = x$

$\Rightarrow f(0) = 2 \qquad \Rightarrow f^{-1}(2) = 0$

$f'(f^{-1}(x))\,.\,(f^{-1}(x))' = 1$

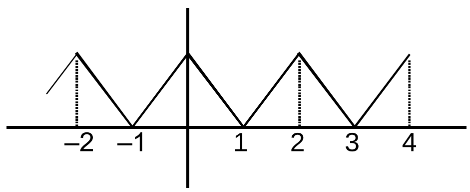
$\Rightarrow (f^{-1}(2))' = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(0)} = \frac{1}{3}$

From (1)

$e^{-x}(f'(x) - f(x)) = \sqrt{x^4+1}$

put x = 0 $\quad f'(0) - 2 = 1 \Rightarrow f'(0) = 3$

(d) $\quad f(x) = \begin{cases} x-1 & 1 \le x < 2 \\ 1-x & 0 \le x < 1 \end{cases}$



f(x) is periodic with period 2

$$I = \int_{-10}^{10} f(x)\cos\pi x\, dx = 2\int_{0}^{10} f(x)\cos\pi x\, dx$$

$$= 10\int_0^2 f(x)\cos\pi x\, dx$$

$$= 10\left[\int_0^1 (1-x)\cos\pi x dx + \int_1^2 (x-1)\cos\pi x dx\right] = 10I_1 + 10I_2$$

$$I_2 = \int_1^2 (x-1)\cos\pi x dx \text{ Put } x - 1 = t = -\int_0^1 x\cos\pi x\, dx$$

$$I_1 = \int_0^1 (1-x)\cos\pi x\, dx = -\int_0^1 x\cos(\pi x)\, dx$$

$$I = 10\left[-2\int_0^1 x\cos\pi x\, dx\right]$$

$$= -20\left[\frac{x\sin\pi x}{\pi} + \frac{\cos\pi x}{x^2}\right]_0^1 = -20\left[-\frac{1}{x^2} - \frac{1}{x^2}\right] = \frac{40}{\pi^2}$$

$$\frac{\pi^2}{10} \times I = 4$$

Sol.12 put $x^2 = t$

$$I = \frac{1}{2}\int_{\ell n 2}^{\ell n 3} \frac{\sin t\, dt}{\sin t + \sin(\ell n 6 - t)} \qquad .....(i)$$

use king property

$$I = \frac{1}{2}\int_{\ell n 2}^{\ell n 3} \frac{\sin(\ell n 6 - t)}{\sin(\ell n 6 - t) + \sin t} dt \qquad .....(ii)$$

Add (i) & (ii)

$$2I = \frac{1}{2}\int_{\ell n 2}^{\ell n 3} \frac{\sin t + \sin(\ell n 6 - t)}{\sin t + \sin(\ell n 6 - t)} dt = \frac{1}{2}\int_{\ell n 2}^{\ell n 3} dt$$

$$2I = \frac{1}{2}[t]_{\ell n 2}^{\ell n 3} \Rightarrow I = \frac{1}{4}(\ell n 3 - \ell n 2)$$

$$I = \frac{1}{4}\ell n\left(\frac{3}{2}\right)$$

Sol.13 $6\int_1^x f(t)\, dt = 3xf(x) - x^3$

diff. both the side w.r.t x

$6f(x).\ 1 = 3[xf'(x) + f(x)] - 3x^2$

$2f(x) = xf'(x) + f(x) - x^2$

$xf'(x) - f(x) = x^2$

$$x\frac{dy}{dx} - y = x^2$$

$$x\frac{dy}{dx} - y = x^2$$

$$\frac{dy}{dx} - \frac{y}{x} = x$$

Here $P = -\dfrac{1}{x}$

Q = x

$$\text{I.F.} = e^{\int P dx} = e^{\int -\frac{1}{x}dx} = e^{-\ell n\, x} = \frac{1}{x}$$

Hence the required solution is

$$\frac{y}{x} = \int x \times \frac{1}{x} dx \Rightarrow \frac{y}{x} = x + c$$

$f(x) = x^2 + cx \quad \because f(1) = 2$

$\Rightarrow 2 = 1 + c$

c = 1

f(2) = 4 + 2 = 6

Sol.14 $$\int_{-\pi/2}^{\pi/2} x^2.\cos x + \int_{-\pi/2}^{\pi/2} \ln\frac{\pi + x}{\pi - x}.\cos x\ dx$$

odd function = 0

$$\int_{-\pi/2}^{\pi/2} x^2.\cos x\, dx$$

Solve by parts

$$\frac{\pi^2}{2} - 4$$